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Research Article

On the PU-Pettis Integral in Banach Space

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ABSTRACT

A Riemannian approach of the PU integral is a Henstock-type method anchored in the concept of a partition of unity. An integral of Pettis type, on the other hand, is essential, somehow, in formulating an integral in a Banach space. In this paper, the PU-Pettis integral is formulated, along with some of its basic properties.

Keywords: Pettis integration, partition of unity, Banach space, linear functionals

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1. INTRODUCTION

A Henstock integration that is endowed with a covering system obtained from a partition of unity is, apparently, the PU-Integral. Up to date, there are relatively few papers pursued with regards to the PU-integral. The definition were first introduced by J. Kurzweil and J. Jarnik and they mentioned that the PU-integral can be untilized in formulating a gauge type integral defined on a smooth manifold. On one hand, Pettis integral started with being an extension of the Lebesgue integral to Banach-valued functions on a measure space. Similar approach of the Pettis integral in some gauge integrals, like the Henstock integral and the McShance integral, were studied. Thus, it is worthwhile, however, to investigate Pettis approach of the PU-integral.

Throughout the rest of the paper, with respect to the perspective of Henstock, the following conventions will be adopted:

1. A finite collection of point-interval pair $\{(t_i, I_i)\}_{i=1}^m$, where I_i is a compact interval, is of Perron type if $t_i \in I_i$ for all $i \leq m$;
2. a gauge δ is any positive functions;
3. for a gauge δ on $[a,b]$, a finite collection of point interval pairs, of Perron type, $\{(t_i, I_i)\}_{i=1}^m$ is said to be δ-fine Perron partition of $[a,b]$ if I_i is a partition of $[a,b]$ and $I_i \subseteq B(t_i, \delta(t_i))$; and

Recently, Boonpogkrong, revisited the PUL integral and it's utilzation for the integrals of a function defined on a smooth manifold. Moreover, Flores and Benitez [4, 5] extended the latter in Stieltjes approach and provided some convergence theorems. In addition, Flores [3] introduced the PUL* integral to a Banach-valued function as an extension of the PUL-integral.

1.1 Review of Literature

Throughout the discussions on the literature, we denote a compact interval in $\mathbb{R}^n$ by $[\mathbf{a},\mathbf{b}] = \prod_{k=1}^n [a_k, b_k]$ with $[a_k, b_k] \subseteq \mathbb{R}$ for each $k = 1,2,\ldots,n$ and $\mu([\mathbf{a},\mathbf{b}]) = \prod_{k=1}^n (b_k - a_k)$ be the volume of $[\mathbf{a},\mathbf{b}]$. In addition, $\mathbb{R}^n$ is equipped with the norm $\|\cdot\|$ defined by

$$\| \mathbf{x} \| = \max\{|x_i| : i = 1,2,\cdots,n\}$$

and for $r > 0$, we write $B(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\|_n < r\}$, where

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n)$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

For a smooth function $\psi: [\mathbf{a},\mathbf{b}] \to \mathbb{R}$, the *support* of ψ, denoted by suppψ, is given by

$$\text{supp}\psi = \overline{\{\mathbf{x} \in [\mathbf{a},\mathbf{b}] : \psi(\mathbf{x}) \neq 0\}},$$

where $\overline{A}$ denotes the closure of $A \subseteq \mathbb{R}^n$. A *gauge* on $[\mathbf{a},\mathbf{b}]$ is a positive function defined on $[\mathbf{a},\mathbf{b}]$.

Definition 3.1 *[3]* A finite collection $\{\psi_k\}_{k=1}^m$ of smooth functions defined on $[\mathbf{a},\mathbf{b}]$ is said to be a *partial partition of unity* if the following holds:

1. $\psi_k(\boldsymbol{\xi}) \geq 0$ for almost all $\boldsymbol{\xi} \in [\mathbf{a},\mathbf{b}]$ and for all $k \in \{1,2,\cdots,m\}$ and
2. $\sum_{k=1}^m \psi_k(\boldsymbol{\xi}) \leq 1$ for almost all $\boldsymbol{\xi} \in [\mathbf{a},\mathbf{b}]$.

If $\sum_{k=1}^m \psi_k = 1$ a.e. on $[\mathbf{a},\mathbf{b}]$, then $\{\psi_k\}_{k=1}^m$ is said to be a *partition of unity*.

Definition 3.2 *[3]* Let $\psi: [\mathbf{a},\mathbf{b}] \to \mathbb{R}$ be a smooth function and δ a gauge on $[\mathbf{a},\mathbf{b}]$. A triple $(\boldsymbol{\xi}, \mathbf{I}, \psi)$, with $\boldsymbol{\xi} \in \mathbf{I} \subseteq [\mathbf{a},\mathbf{b}]$, is said to be *$\delta$-fine* if

$$\text{supp}\psi \subseteq \mathbf{I} \subseteq B(\boldsymbol{\xi}; \delta(\boldsymbol{\xi})).$$

If δ_1 and δ_2 are gauges on $[\mathbf{a},\mathbf{b}]$ such that $\delta_1(\boldsymbol{\xi}) \geq \delta_2(\boldsymbol{\xi})$ and $(\boldsymbol{\xi}, \mathbf{I}, \psi)$ is δ_2-fine, then $(\boldsymbol{\xi}, \mathbf{I}, \psi)$ is also δ_1-fine.

Throughout this paper, a *division* of $[\mathbf{a},\mathbf{b}]$ is a finite collection $D = \{\mathbf{I}_k\}_{k=1}^m$ of subintervals $\mathbf{I}_k = \prod_{i=1}^n [a_i^{(k)}, b_i^{(k)}]$ of $[\mathbf{a},\mathbf{b}]$ such that $int(\mathbf{I}_k) \cap int(\mathbf{I}_j) = \varnothing$ for $k \neq j$ and $\bigcup_{k=1}^m \mathbf{I}_k = [\mathbf{a},\mathbf{b}]$.

Definition 3.3 [1] A finite collection $D = \{(\boldsymbol{\xi}_k, \mathbf{I}_k, \psi_k)\}_{k=1}^m$ is said to be a *δ-fine partial PU-division* of $[\mathbf{a},\mathbf{b}]$ if the collection $\{\psi_k\}_{k=1}^m$ is a partial partition of unity and every $(\boldsymbol{\xi}_k, \mathbf{I}_k, \psi_k)$ is δ-fine. If $\{\psi_k\}_{k=1}^m$ is a partition of unity, then D is said to be a *δ-fine PU-division* of $[\mathbf{a},\mathbf{b}]$.

The existence of δ-fine divisions of $[\mathbf{a},\mathbf{b}]$ is guaranteed by the open covering theorem and the existence of a partition of unity [1].

Let $D = \{(\boldsymbol{\xi}, \mathbf{I}, \psi)\}$ is a finite collection of δ-fine PU division of $[\mathbf{a},\mathbf{b}]$. We define

$$S(f, D) = \sum_D f(\boldsymbol{\xi}) \int_{\mathbf{I}} \psi,$$

where $\int_{\mathbf{I}} \psi$ is the Lebesgue integral of ψ on $\mathbf{I}$.

Remark 3.4 [4] If $D_1 = \{(\boldsymbol{\xi}_k, \mathbf{I}_k, \varphi_k)\}_{k=1}^{m}$ and $D_2 = \{(\boldsymbol{\xi}_j, \mathbf{I}_j, \psi_j)\}_{j=1}^{m}$ are two δ-fine PU-divisions of $[\mathbf{a}, \mathbf{b}]$, then $S(f, D_1) = S(f, D_2)$.

1.2. Objectives

The goal of this paper is to formulate the PU-Pettis integral in Banach Space including some of its fundamental properties. To attain the above mentioned, the following are the objectives of the study: To define the PUL *-integral in Banach Space and to investigate its simple properties; to duduce the following:

1. Linearity property;
2. Cauchy-Criterion and its corollary results; and
3. The existence theorem;

Definition 4.1 *[3]* Let $(X, \|\cdot\|)$ be a Banach space. A function $f : [\mathbf{a}, \mathbf{b}] \to X$ is said to be *PU integrable to* $A \in X$ over $[\mathbf{a}, \mathbf{b}]$ if for every $\epsilon > 0$, there exists a gauge δ on $[\mathbf{a}, \mathbf{b}]$ such that for every δ-fine PU-division $D = \{(\boldsymbol{\xi}_k, \mathbf{I}_k, \psi_k)\}_{k=1}^{m}$ of $[\mathbf{a}, \mathbf{b}]$, we have

$$\|S(f, D) - A\| < \epsilon.$$

If A is the PU-integral of f over $[\mathbf{a}, \mathbf{b}]$, then we write

$$A = (\mathcal{P}) \int_{[\mathbf{a},\mathbf{b}]} f.$$

Note that Remark 3.4 means that a PUL * sum is independent of the choice of the partition of unity. Consequently, the value the PUL * integral is independent of the choice of the partition of unity.

Example 4.2 Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & if \quad x \in \mathbb{Q} \cap [0,1] \\ 0, & otherwise \end{cases}$$

for all $X \in [0,1]$. Let $\epsilon > 0$. Note that $\mathbb{Q}$ is a countable set; thus, we write $\mathbb{Q} = \{q_n\}_{n=1}^{\infty}$. Define $\delta : [0,1] \to \mathbb{R}$ by

$$\delta(x) = \begin{cases} \frac{\epsilon}{2^{n+1}}, & if \quad x \in \mathbb{Q} \cap [0,1] \\ 1, & otherwise \end{cases}$$

for all $X \in [0,1]$. Here, δ is a gauge on $[0,1]$. Fix $D = \{\xi, I, \varphi\}$, a δ-fine PU-division of $[0,1]$. Observe that

$$|\sum_D f(\xi) \int_I \varphi - 0| = |\sum_D f(\xi) \int_I \varphi| = |\sum_{\xi \in [0,1] \cap \mathbb{Q}}^{D} f(\xi) \int_I \varphi + \sum_{\xi \in [0,1] \setminus \mathbb{Q}}^{D} f(\xi) \int_I \varphi|$$

$$= |\sum_{\xi \in [0,1] \cap \mathbb{Q}}^{D} f(\xi) \int_I \varphi| = |\sum_{\xi \in [0,1] \cap \mathbb{Q}}^{D} \int_I \varphi| = \sum_{\xi \in [0,1] \cap \mathbb{Q}}^{D} |\int_I \varphi|$$

$$\leq \sum_{\xi \in [0,1] \cap \mathbb{Q}}^{D} \int_I |\varphi| \leq \sum_{\xi \in [0,1] \cap \mathbb{Q}}^{D} \int_I 1 = \sum_{\xi \in [0,1] \cap \mathbb{Q}}^{D} \mu(I)$$

$$< \sum_{\xi \in [0,1] \cap \mathbb{Q}}^{D} \delta(\xi) < \sum_{n=1}^{\infty} \delta(\xi) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}}$$

$$= \epsilon,$$

where μ is the $Lebesgue$ measure. This means that f, the $Dirichlet$ function, also called as the $Weierstrass$ function, is PUL^*-integrable to 0 on [0,1].

Recall that the Dirichlet function fails to be Riemann integrable; hence the latter example portays an important facet of the PUL^*-integral.

2. PROCEDURE AND METHODOLOGY

Researches in pure mathematics in nature are basic research. Thus, procedures and methodologies endowed in the study are straightforward; namely, gathering of reading materials such as books, monographs, and research articles that is related and substantial to the results obtained in the study. Results are classified as either Theorems (main results), Propositions, Lemmas, Corollaries or as immediate as Remarks. These results are to be verified only, with rigor and rigid manner, by valid proofs.

3. RESULTS AND DISCUSSION

In this section, the main results of the paper will be presented incuding all of its proofs and discussions. For simplicity, provided that it is free from confusion, we denote a compact interval in $[\mathbf{a}, \mathbf{b}]$ by $\mathbf{I}$.

Throughout the rest of the paper, provided these are free from confusions, we denote by $\mathbf{I}, \mathbf{J}, \cdots$ a compact interval in $\mathbb{R}^n$, X is a Banach space, and for an X-valued function f defined on $\mathbf{I}$, the PUL *-integral of f over $\mathbf{I}$ will be denoted by

$$(\mathcal{P}) \int_{\mathbf{I}} f.$$

Proposition 6.1 *Suppose that $f: \boldsymbol{I} \to X$ is PUL***-integrable over $\boldsymbol{I}$. Then for every $x^* \in X^*$, $x^*: \boldsymbol{I} \to \mathbb{R}$ is PUL***-integrable over $\boldsymbol{I}$ and*

$$(\mathcal{P}) \int_{\mathbf{I}} x^*(f) = x^*((\mathcal{P}) \int_{\mathbf{I}} f).$$

Proof. Assume that f is PUL*-integrable to $A \in X$ over $\mathbf{I}$. Let $x^* \in X^*$ and let $\epsilon > 0$. Then there exists $\delta_\epsilon: \mathbf{I} \to \mathbb{R}^+$ such that

$$\| S(f, D) - A \|_X < \frac{\epsilon}{\|x^*\|_{X^*} + 1}$$

for every δ_ϵ-fine PU-division of $\mathbf{I}$. Fix a δ_ϵ-fine division D of $\mathbf{I}$. Then observe that

$$\| S(x^* f, D) - x^*((\mathcal{P}) \int_{\mathbf{I}} f) \|_X = \| x^*(S(f, D) - (\mathcal{P}) \int_{\mathbf{I}} f) \|_X$$

$$\leq \| x^* \|_{X^*} \| S(f, D) - (\mathcal{P}) \int_I f \|_X$$

$$< (\| x^* \|_{X^*} + 1) \cdot \frac{\epsilon}{\|x^*\|_{X^*} + 1}$$

$$= \epsilon;$$

and the conclusion follows. □

Definition 6.2 *Let $f: \boldsymbol{I} \to X$ be any X-valued function such that $x^*(f): \boldsymbol{I} \to \mathbb{R}$ is PUL***-integrable over $\boldsymbol{I}$ for all $x^* \in X^*$. If for every interval $\boldsymbol{J} \subseteq \boldsymbol{I}$, there exists an element $x_J^{**} \in X^{**}$ such that*

$$x_{\mathbf{J}}^{**}(x^*) = (\mathcal{P}) \int_{\mathbf{J}} x^*(f)$$

for all $x^* \in X^*$, then f is called the *PUL***-Dunford integrable* over $\mathbf{I}$. Moreover, for $\mathbf{J} \subseteq \mathbf{I}$, we write the PUL*-Dunford integral of f over $\mathbf{J}$ by

$$(\mathcal{PD}) \int_{\mathbf{J}} f = x_{\mathbf{J}}^{**} \in X^{**}.$$

Here, we denote by $\mathcal{PD}$ the set of all PUL*-Dunford integrable functions $f: \mathbf{I} \to \mathbb{R}$.

Remark 6.3 *Let $f: \boldsymbol{I} \to X$ be an X-valued function in $\mathcal{PD}(\boldsymbol{I})$. If $\boldsymbol{J} \subseteq \boldsymbol{I}$, then*

$$(\mathcal{PD}) \int_{\mathbf{J}} f = (\mathcal{PD}) \int_{\mathbf{I}} f \cdot \chi_{\mathbf{J}}.$$

Theorem 6.4 *The PUL***-Dunford integral of f over $\boldsymbol{I}$, if it exists, is unique.*

Proof. Suppose that f is PUL*-Dunford integrable over $\mathbf{I}$. Then for every $x^* \in X^*$, $x^*(f)$ is PUL*-integrable and for each $\mathbf{J} \subseteq \mathbf{I}$, there exists $x_{\mathbf{J}}^{**} \in X^{**}$ such that

$$x_{\mathbf{J}}^{**}(x^*) = (\mathcal{P}) \int_{\mathbf{J}} x^*(f)$$

for all $x^* \in X^*$. For $\mathbf{J} \subseteq \mathbf{I}$, assume that $x_{\mathbf{J}}^{**}, y_{\mathbf{J}}^{**} \in X^{**}$ are the values of the PUL*-Dunford integral of f over $\mathbf{I}$. Let $x^* \in X^*$. Then $x^*(f)$ is PUL*-integrable over $\mathbf{I}$. But

$$x_{\mathbf{J}}^{**}(x^*) = (\mathcal{P}) \int_{\mathbf{J}} x^*(f) = y_{\mathbf{J}}^{**}(x^*). □$$

Theorem 6.5 *Let $f, g \in \mathcal{PD}(\boldsymbol{I})$. Then for each $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in \mathcal{PD}(\boldsymbol{I})$ and*

$$(\mathcal{PD}) \int_{\mathbf{J}} (\alpha f + \beta g) = \alpha \cdot (\mathcal{PD}) \int_{\mathbf{J}} f + \beta \cdot (\mathcal{PD}) \int_{\mathbf{J}} g$$

for all $\mathbf{J} \subseteq \mathbf{I}$.

Proof. Let $\alpha, \beta \in \mathbb{R}$. Fix $x^* \in X^*$. Observe that

$$x^*(\alpha f + \beta g) = \alpha \cdot x^*(f) + \beta \cdot x^*(g)$$

where $\alpha \cdot x^*(f) + \alpha \cdot x^*(f) + \beta \cdot x^*(g)$ is PUL*-integrable over $\mathbf{I}$ and

$$(\mathcal{P}) \int_{\mathbf{I}} [\alpha x^*(f) + \beta x^*(g)] = \alpha \cdot (\mathcal{P}) \int_{\mathbf{I}} f + \beta \cdot (\mathcal{P}) \int_{\mathbf{I}} g. \quad (6.1)$$

Next, let $\mathbf{J} \subseteq \mathbf{I}$. By the integrability, in a sense of PUL*-Dundord, of f and g over $\mathbf{I}$, we choose the operators $x_{\mathbf{J}}^{**}, y_{\mathbf{J}}^{**} \in X^{**}$ such that for each $x^* \in X^*$

$$x_{\mathbf{J}}^{**}(x^*) = (\mathcal{P}) \int_{\mathbf{J}} x^*(f)$$

and

$$y_{\mathbf{J}}^{**}(x^*) = (\mathcal{P}) \int_{\mathbf{J}} y^*(f).$$

Observe that $\alpha x_{\mathbf{J}}^{**} + \beta y_{\mathbf{J}}^{**} \in X^{**}$. Put $z_{\mathbf{J}}^{**} = \alpha x_{\mathbf{J}}^{**} + \beta y_{\mathbf{J}}^{**}$. Fix $x^* \in X^*$. Then by (6.1)

$$z_{\mathbf{J}}^{**}(x^*) = (\alpha x_{\mathbf{J}}^{**} + \beta y_{\mathbf{J}}^{**})(x^*) = \alpha x_{\mathbf{J}}^{**}(x^*) + \beta y_{\mathbf{J}}^{**}(x^*)$$

$$= \alpha \cdot x_{\mathbf{J}}^{**}(x^*) + \beta \cdot y_{\mathbf{J}}^{**}(x^*)$$

$$= \alpha \cdot (\mathcal{P}) \int_{\mathbf{J}} x^*(f) + \beta \cdot (\mathcal{P}) \int_{\mathbf{J}} x^*(g)$$

$$= (\mathcal{P}) \int_{\mathbf{J}} [\alpha x^*(f) + \beta x^*(g)]$$

$$= (\mathcal{P}) \int_{\mathbf{J}} x^*(\alpha f + \beta g).$$

This means that that $\alpha f + \beta g$ is PUL*-Dunford integrable. Moreover,

$$(\mathcal{PD}) \int_{\mathbf{J}} (\alpha f + \beta g) = z_{\mathbf{J}}^{**} = \alpha \cdot x_{\mathbf{J}}^{**} + \beta \cdot y_{\mathbf{J}}^{**}$$

$$= \alpha \cdot (\mathcal{PD}) \int_{\mathbf{J}} x * (f) + \beta \cdot (\mathcal{PD}) \int_{\mathbf{J}} x * (g);$$

and the proof follows. ∎

We now define the PUL*-Pettis integral of $f: \mathbf{I} \to X$ and such integral of f over the closed and bounded interval $\mathbf{J} \subseteq \mathbf{I}$, if it exists, will be denoted by

$$(\mathcal{PP}) = \int_{\mathbf{J}} f.$$

Definition 6.6 *If $f: \boldsymbol{I} \to X$ is PUL*-Dunford integrable where $\int_J f \in X$ for every compact interval $\boldsymbol{J} \subseteq \boldsymbol{I}$, then f is said to be PUL*-Pettis integrable over $\boldsymbol{J}$ and*

$$(\mathcal{PD}) \int_{\mathbf{J}} f = (\mathcal{PP}) \int_{\mathbf{J}} f$$

is called the *PUL*-Pettis integral of f over $\boldsymbol{J} \subseteq \boldsymbol{I}$*. We denote by $\mathcal{PP}$ the set of all PUL*-Pettis integrable functions $f: \mathbf{I} \to X$.

Remark 6.7 *If $(\mathcal{PP}) \int_J f$ exists, then $(\mathcal{PP}) \int_J f \in e(X) \subseteq X^{**}$, where e is the canonical embedding of X into X^{**}.*

Throughout the rest of the paper, if no confussion arises, e means the canonical mapping from $X \to X^{**}$.

Theorem 6.8 *The PUL*-Pettis integral of $f: \boldsymbol{I} \to X$, if it exists, is unique.*

Proof. Assume that $(\mathcal{PP}) \int_{\mathbf{J}} f$ exists for all $\mathbf{J} \subseteq \mathbf{I}$. By Definition 6.6,

$$(\mathcal{PP}) \int_{\mathbf{J}} f = (\mathcal{PD}) \int_{\mathbf{J}} f.$$

Fix $\mathbf{J} \subseteq \mathbf{I}$. Since $(\mathcal{PD}) \int_{\mathbf{J}}$ is unique; then so as $(\mathcal{PP}) \int_{\mathbf{J}} f$. ∎

Theorem 6.9 *Let $f, g \in \mathcal{PP}$. Then for each $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in \mathcal{PP}$ and*

$$(\mathcal{PP}) \int_{\mathbf{J}} (\alpha f + \beta g) = \alpha \cdot (\mathcal{PP}) \int_{\mathbf{J}} f + \beta \cdot (\mathcal{PP}) \int_{\mathbf{J}} g$$

for all $\mathbf{J} \subseteq \mathbf{I}$.

Proof. Let $\alpha, \beta \in \mathbb{R}$ and let $\mathbf{J} \subseteq \mathbf{I}$. Then $\alpha \cdot f + \beta \cdot g \in \mathcal{PD}$ and

$$(\mathcal{PD}) \int_{\mathbf{J}} [\alpha \cdot f + \beta \cdot g] = \alpha \cdot (\mathcal{PD}) \int_{\mathbf{J}} f + \beta \cdot (\mathcal{PD}) \int_{\mathbf{J}} g. (*)$$

Since $(\mathcal{PD}) \int_{\mathbf{J}} f, (\mathcal{PD}) \int_{\mathbf{J}} g \in e(X)$, we choose $x_1, x_2 \in X$ such that

$$(\mathcal{PD}) \int_{\mathbf{J}} f = x_1$$

and

$$(\mathcal{PD}) \int_{\mathbf{J}} g = x_2.$$

So, $\alpha \cdot x_1 + \beta \cdot x_2 \in X$. Note that $\alpha \cdot f + \beta \cdot g \in \mathcal{PD}$. By $(*)$,

$$\begin{aligned}(\mathcal{PD}) \int_{\mathbf{J}} [\alpha \cdot f + \beta \cdot g] &= \alpha \cdot (\mathcal{PD}) \int_{\mathbf{J}} f + \beta \cdot (\mathcal{PD}) \int_{\mathbf{J}} g \\ &= \alpha \cdot e(x_1) + \beta \cdot e(x_2) \\ &= e(\alpha \cdot x_1 + \beta \cdot x_2) \\ &\in e(X).\end{aligned}$$

This means that $\alpha \cdot f + \beta \cdot g \in \mathcal{PP}$ and

$$\begin{aligned}(\mathcal{PP}) \int_{\mathbf{J}} [\alpha \cdot f + \beta \cdot g] &= (\mathcal{PD}) \int_{\mathbf{J}} [\alpha \cdot f + \beta \cdot g] \\ &= \alpha \cdot (\mathcal{PD}) \int_{\mathbf{J}} f + \beta \cdot (\mathcal{PD}) \int_{\mathbf{J}} g \\ &= \alpha \cdot (\mathcal{PP}) \int_{\mathbf{J}} f + \beta \cdot (\mathcal{PP}) \int_{\mathbf{J}} g;\end{aligned}$$

and the conclusion follows. ∎

Theorem 6.10 *A function $f: \boldsymbol{I} \to X$ is PUL*-Dunford integrable if an only if $x^*(f)$ is PUL*-integrable over $\boldsymbol{I}$ for all $x^* \in X^*$.*

Proof. If $f: \mathbf{I} \to \mathbb{R}$ is PUL*-Dunford integrable over $\mathbf{I}$, then by Definition 6.2, $x^*(f): \mathbf{I} \to \mathbb{R}$ is PUL*-integrable over $\mathbf{I}$. Conversely, suppose that $x^*(f): \mathbf{I} \to \mathbb{R}$ is PUL*-integrable on $\mathbf{I}$. Let $\mathbf{J} \subseteq \mathbf{I}$. Then $x^*(f)$ is PUL*-integrable over $\mathbf{J}$. Thus, $(\mathcal{P}) \int_{\mathbf{J}} x^*(f) \in \mathbb{R}$. Define $x^{**}: X^* \to \mathbb{R}$ by

$$x_{\mathbf{J}}^{**}(x^*) = (\mathcal{P}) \int_{\mathbf{J}} x^* f, \qquad for\ all\ \ x^* \in X^*.$$

Then $x_{\mathbf{J}}^{**} \in X^{**}$. Therefore, f is PUL*-Dunford integrable over $\mathbf{I}$. ∎

Remark 6.11 (Additivity of the PUL*-Dunford) *Let $f: \boldsymbol{I} \to X$ is PUL*-Dunford integrable over $\boldsymbol{I}$. If f is PUL*-Dunford integrable over the closed and bounded subintervals $\boldsymbol{J}, \boldsymbol{K}$ of $\boldsymbol{I}$ where $\boldsymbol{J}$ and $\boldsymbol{K}$ nonoverlapping, then f is PUL*-Dunford integrable over $\boldsymbol{J} \cup \boldsymbol{K}$ and*

$$(\mathcal{PD}) \int_{\mathbf{I} \cup \mathbf{J}} f = (\mathcal{PD}) \int_{\mathbf{J}} f + (\mathcal{PD}) \int_{\mathbf{K}} f.$$

Remark 6.12 (Additivity of the PUL*-Pettis) *Let $f: \boldsymbol{I} \to X$ is PUL*-Pettis integrable over $\boldsymbol{I}$. If f is PUL*-Pettis integrable over the closed and bounded subintervals*

$\boldsymbol{J}, \boldsymbol{K}$ *of* $\boldsymbol{I}$ *where* $\boldsymbol{J}$ *and* $\boldsymbol{K}$ *nonoverlapping, then* f *is PUL***-Pettis integrable over* $\boldsymbol{J} \cup \boldsymbol{K}$ *and*

$$(\mathcal{PP}) \int_{\mathbf{I} \cup \mathbf{J}} f = (\mathcal{PP}) \int_{\mathbf{J}} f + (\mathcal{PP}) \int_{\mathbf{K}} f.$$

Theorem 6.13 (Cauchy Criterion) *A function* $f: \boldsymbol{I} \to \mathbb{R}$ *is PUL***-Dunford integrable over* $\boldsymbol{I}$ *if and only if for every* $\epsilon > 0$, *there exists a gauge* $\delta: \boldsymbol{I} \to \mathbb{R}^+$ *such that if* D_1 *and* D_2 *are two* δ*-fine divisions of* $\boldsymbol{I}$, *then*

$$\| S(f, D_1) - S(f, D_2) \|_X < \epsilon.$$

Proof: Let $f: \mathbf{I} \to \mathbb{R}$ be PUL*-Dunford integrable over $\mathbf{I}$. Fix $\epsilon > 0$. By Theorem 6.10, $x^*(f)$ is PUL*-integrable over $\mathbf{I}$ for all $x^* \in X^*$. Here, we are done if $x^* \in X^*$ is the zero map. Assume that $x^* \in X^* \setminus \{\theta\}$, where θ is the zero map. Then we choose $\delta: \mathbf{I} \to \mathbb{R}^+$ such that for any two δ-fine divisions D_1 and D_2 of $\mathbf{I}$, we have

$$\| S(x^*(f), D_1) - S(x^*(f), D_2) \|_X < \epsilon \cdot \| x^* \|_{X^*},$$

that is,

$$\| S(f, D_1) - S(f, D_2) \|_X = \| S(x^*(f), D_1) - S(x^*(f), D_2) \| \cdot \frac{1}{\|x^*\|_{X^*}}$$

$$< \frac{\epsilon \cdot \|x^*\|_{X^*}}{\|x^*\|_{X^*}}$$

$$= \epsilon.$$

Author Contributions: G. Flores conceptualized the study, developed the methodology, and supervised the research process. A. Flores also performed the formal analysis, interpreted the results, and wrote the original draft of the manuscript specifically on the algebraic structures endowed in the study. All authors contributed to the review and editing of the manuscript, approved the final version for publication, and agree to be accountable for all aspects of the work.

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