Central Mindanao University
Journal of Science
ISSN Print: 0116-7847 ISSN Online: 2704-3703

Quantum Description of a Damped Coupled Harmonic Oscillator via White-Noise Analysis

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ABSTRACT

In this paper, the quantum mechanical dynamics of a particle subjected to a damped coupled harmonic oscillator potential was investigated by solving its quantum propagator using the Hida-Streit formulation—also known as the White-Noise analysis. A coordinate transformation to decouple the system was also performed. After the decoupling process, the authors obtained a separate expression of the Lagrangian for a one-dimensional damped harmonic oscillator. Then, the obtained Lagrangian was cast to the classical action and evaluated their propagator using the white noise path integration. The full form of the propagator was solved by taking the product of the individual propagator, and from that, the wave function, particularly the ground state wave function was extracted by symmetrization and setting the quantum number $n_1 = n_2 = 0$. The result agrees with the propagator of a coupled harmonic oscillator without damping (Pabalay et.al, 2007) as the damping factor γ is turned off.

Keywords: Propagator, White-Noise Analysis, Coupled Oscillators, Quantum Mechanics

INTRODUCTION

Due to its wide application in understanding the dynamics of the universe, studies in oscillatory motion had been dramatically growing throughout the years. It is viewed from the fact that oscillations are present in all systems in nature—from microscopic to macroscopic point of views. There are classifications of an oscillatory motion; we have simple harmonic, damped harmonic, coupled harmonic oscillations, among others. These systems have many interesting applications, for example, in the context of the general theory of relativity, it was shown in (Ben-Aryeh, 2008) that the general theory of time-dependent harmonic oscillator—harmonic oscillator with time-dependent frequency and time-dependent mass— is applied for studying certain quantum effects in the interferometers for detecting gravitational waves.

In this article, we will study the coupled harmonic oscillator system with isotropic mass and time-independent frequency. Coupled harmonic oscillation is close to a non-ideal system, for it represents a natural system for two or more particles interacting with each other and exhibits a transport of energy due to their interaction. Many physical phenomena illustrate coupled oscillations. For instance, when a solid material was subjected to extreme temperature, the electron in the material oscillates rapidly, which causes the electron to interact with another electron such that the individual movements of each molecule are now influenced by the other molecule. Hence, they are coupled.

The system of coupled oscillations has been extensively studied by many authors in the past decade (de Souza Dutra, 1992) and until this very moment (Pabalay et.al, 2007; Rangaig et.al, 2017; Butanas et.al, 2016; Gallo et.al, 2019). Aside from the fact that it is an interesting system to deal with, there are many physical situations wherein the principles of coupled oscillations can be applied. For instance, in meteorology, in 2017 (Muraki, 2017), a coupled harmonic oscillator was used to model the solar activity and El Niño-Southern Oscillation (ENSO) in Japan. Also, coupled oscillators are applied in the context of coupled semiconductor lasers (Dente et.al, 1990), in which the system is subjected to additional effects such as time delays which arises by introducing additional degrees of freedom to the system.

The quantum representation of the coupled harmonic oscillator system has already been investigated. See for example the paper of Pabalay and Bornales (Pabalay et.al, 2007) wherein they obtained an expression for the quantum mechanical propagator for a coupled harmonic oscillator with coupling constant λ; Rangaig et al. (Rangaig et.al, 2017) in which they solved the quantum dynamic propagator of a harmonic oscillator with the uniform electric field in N-multimode harmonic oscillator bath; Dutra (de Souza Dutra, 1992) where the quantum propagator for a driven coupled harmonic oscillator is solved; and many others (Butanas et.al, 2017; Macedo et.al, 2012; Bernido et.al, 2012).

In solving the quantum propagator, one way is by path integration introduced by Feynman, in which the sum of over-all possible trajectories of a particle is taken. However, even though it describes the system correctly, the formulation is mathematically ill for its lack of rigor. Fortunately, there is a method that casts the Feynman path integral formulation to a more rigorous approach, and here comes the Hida-Streit formulation also known as white-noise path integration. This formulation has been successful in solving many quantum systems (Butanas

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Received 28th August 2020; Accepted 12th January 2021

et.al, 2017).

Here in this paper, the authors describe a specific coupled oscillatory motion in which an additional parameter known as the damping parameter γ was introduced. The damping alters the behavior of a coupled harmonic oscillator considered in (Pabalay et.al, 2007). It can be thought of as friction or a dissipative force applied to a coupled harmonic oscillator with coupling constant λ. The authors want to obtain its quantum propagator using the Hida-Streit approach.

METHODOLOGY

Decoupling Procedure

The Lagrangian of the damped coupled harmonic oscillator is given by

$$L = \left[\frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}m\Omega^2(x_1^2 + x_2^2) - \lambda x_1 x_2\right] e^{\gamma t} \quad (1)$$

where λ is the generalized coupling factor treated as constant, x_1 and x_2 are the generalized coordinates, m is the mass, Ω is the frequency, and γ is the damping factor. The heart of this paper is mainly due to the introduction of the damping parameter. The main goal here is to decouple the Lagrangian L given in Eq. (1). One can decouple L by performing a coordinate transformation given by

$$\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}. \quad (2)$$

Using equation (2), one can rewrite equation (1) as

$$L_T = \left[\frac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2) - \frac{1}{2}m\Omega^2(y_1^2 + y_2^2) - \lambda\left(\frac{y_2^2 - y_1^2}{2}\sin 2\phi + y_1 y_2 \cos 2\phi\right)\right] e^{\gamma t}. \quad (3)$$

Equation (3) has coupling in y coordinates and there is a need to eliminate it. With that, one needs to impose the condition,

$$y_1 y_2 \cos\phi = 0\,, \quad (4)$$

which gives us the relation given by

$$\phi = \left(\frac{n}{2} + \frac{1}{4}\right)\pi. \quad (5)$$

Using equations (4) and (5), one can rewrite the transformed Lagrangian L_T as follows

$$L_T = \left(\frac{1}{2}m\dot{y}_1^2 - \frac{1}{2}m\Omega_1^2 y_1^2\right) e^{\gamma t} + \left(\frac{1}{2}m\dot{y}_2^2 - \frac{1}{2}m\Omega_2^2 y_2^2\right) e^{\gamma t} \quad (6)$$

where

$$\Omega_1^2 = \Omega^2 - \frac{\lambda}{m} \quad (7)$$

$$\Omega_2^2 = \Omega^2 + \frac{\lambda}{m}. \quad (8)$$

Notice that from equation (6), we can write $L_T = L_{T1} + L_{T2}$ where

$$L_{T1} = \left(\frac{1}{2}m\dot{y}_1^2 - \frac{1}{2}m\Omega_1^2 y_1^2\right) e^{\gamma t} \quad (9)$$

$$L_{T2} = \left(\frac{1}{2}m\dot{y}_2^2 - \frac{1}{2}m\Omega_2^2 y_2^2\right) e^{\gamma t}. \quad (10)$$

At this point, the problem is reduced into two damped harmonic oscillators.

Brief review on the Hida-Streit approach

In this section, some of the main points in using the Hida-Streit formulation will be reviewed to solve the quantum mechanical propagator of the system.

Describing the system's propagator in the quantum mechanics regime amounts to solving the Feynman path integral given by

$$K(x', x''; t', t'') = \int D[x] e^{(iS/\hbar)} \quad (11)$$

where D[x] is the Lebesgue measure and S is the classical action expressed in terms of the Lagrangian given by,

$$S = \int L\, dt. \quad (12)$$

Now, to recast the Feynman path integral in the context of white-noise analysis, one needs to perform the following main steps:

- First, parametrize the particle's path in terms of the Brownian motion $B(t)$, given by

$$x(t) = x_o + \sqrt{\frac{\hbar}{m}} B(t) \quad (13)$$

- Second, take the correspondence between the Lebesgue $D[x]$ and Gaussian measure $d\mu\ \omega$

$$D[x] = \exp\left[\frac{1}{2}\int \omega^2(t) dt\right] d_\mu \omega. \quad (14)$$

- Last, fix the endpoint of the particle's path by introducing a Donsker delta function $\delta(x(t) - x1)$, where $x(t)$ is the parametrized path of the particle.

After performing these main steps, one must express the path integral in terms of the white noise variable ω and then take the T-transform (Bernido et.al, 2012) and solve for the propagator.

The Propagator of a damped harmonic oscillator

The propagator of a damped harmonic oscillator was solved by (Cubero et.al, 2010) using white noise analysis. In this section, the authors utilize it to solve the propagator of a damped coupled harmonic oscillator. In our case, the Lagrangian is given in equations (9) and (10). Now, the parametrization of the path of the particle is given by,

$$y_1 = \sqrt{\frac{\hbar}{m\, e^{\gamma t}}} B_1. \tag{15}$$

Incorporating Eq. (15) to Eq. (12), it yields the following expression of the classical action, *S*, given by,

$$S = \frac{\hbar}{2}\left[\int \omega_1^2 dt - \int \Omega_{1\gamma}^2 B_1^2 dt - \int \gamma \omega_1 B_1 dt\right], \tag{16}$$

where

$$B(t) = \int \omega(t) dt. \tag{17}$$

Given the expression for the classical action *S* in Equation (16), the exponential term in equation (11) can be written as,

$$\exp\left(\frac{iS}{\hbar}\right) = \exp\left(\frac{i}{2}\int \omega_1^2 dt - \frac{i\Omega_{1\gamma}^2}{2}\int B_1^2\, dt - \frac{i\gamma}{2}\int B_1\omega_1 dt\right), \tag{18}$$

where

$$\Omega_{1\gamma}^2 = \Omega_1^2 - \frac{\gamma^2}{4}. \tag{19}$$

The first term in the exponent of equation (18) can be written as

$$\frac{i}{2}\int \omega_1^2 dt = \frac{1}{2}\langle \omega_1, i\omega_1 \rangle. \tag{20}$$

Using equation (15), one can rewrite the second term in the exponent of equation (18) a

$$-\frac{i\Omega_{1\gamma}^2}{2}\int B_1^2 dt = -\frac{i}{\hbar}\int V_1(y) dt, \tag{21}$$

where

$$V_1(y) = \frac{m\Omega_{1\gamma}^2 \exp(\gamma t)}{2} y_1^2. \tag{22}$$

Furthermore, equation (21) can be rewritten as

$$-\frac{i\Omega_{1\gamma}^2}{2}\int B_1^2 dt = -\frac{i}{\hbar} S_1, \tag{23}$$

where

$$S_1 = \int \frac{m\Omega_{1\gamma}^2 \exp(\gamma t)}{2} y_1^2 dt. \tag{24}$$

Using Taylor series expansion, equation (24) is expanded to

$$S_1(y) \approx S_1(y_o) + \int \omega_1(\tau)\frac{\delta S_1(y_o)}{\delta\, \omega_1(\tau_1)} d\tau + \frac{1}{2}\int \omega_1(\tau_1)\omega_2(\tau_2)\frac{\delta^2 S_1(y_o)}{\delta\omega_1(\tau_1)\delta\omega_1(\tau_2)} d\tau_1 d\tau_2 + \cdots, \tag{25}$$

and employing the theory of Quadratic Lagrangian, this expansion expands only up to its third term and the rest becomes zero, hence the second term of equation (18) becomes

$$-\frac{i\Omega_{1\gamma}^2}{2}\int B_1^2 dt = -\frac{i}{2\hbar}\int \omega_1(\tau_1)\omega_1(\tau_2) S_1''(y_o) d\tau_1 d\tau_2 = -\frac{1}{2}\langle \omega_1, i\hbar^{-1} S_1''(y_o)\omega_1 \rangle, \tag{26}$$

where

$$S_1''(y_o) = \frac{\hbar}{m\exp(\gamma t)}\int V_1'' dt. \tag{27}$$

Lastly, the third term in the exponent of equation (18) can be re-expressed using direct integration. Letting $u=B_1$, hence, $du=B_1\, dt$. Since $B(t)=\int \omega(t)dt$, then $du=\omega_1(t)dt$, therefore the third term of equation (18) becomes,

$$-\frac{i\gamma}{2}\int B_1\omega_1 dt = -\frac{i\gamma}{4}\left[\frac{m\exp(\gamma t_1)}{\hbar} y_{11}^2\right]. \tag{28}$$

Combining all the re-expressed terms back equation (18), the exponential term of the Feynman path integral reads as,

$$\exp\left(\frac{iS}{\hbar}\right) = \exp\left[-\frac{i\gamma}{4}\left(\frac{m\exp(\gamma t_1)}{\hbar} y_{11}^2\right)\right] \times \exp\left[\frac{1}{2}\langle \omega_1, i\omega_1\rangle - \frac{1}{2}\langle \omega_1, i\hbar^{-1} S_1''(y_o)\omega_1\rangle\right]. \tag{29}$$

Introducing the Donskers delta function and taking the correspondence between the Lebesgue and Gaussian measure, one can write the Feynman path integral in the context of the Hida-Streit formulation as follows

$$K(0, y_1; 0, t_1) = \int D[x] \exp\left(\frac{iS}{\hbar}\right) = \int_{-\infty}^{+\infty} \sqrt{\frac{m\exp(\gamma t_1)}{\hbar}} \exp\left[\frac{1}{2}\langle \omega_1, (i+1)\omega_1\rangle\right] \exp\left[-\frac{1}{2}\langle \omega_1, i\hbar^{-1} S_1''(y_o)\omega_1\rangle\right] \times \exp\left[-\frac{i\gamma}{4}\left(\frac{m\exp(\gamma t_1)}{\hbar} y_{11}^2\right)\right] \delta\left(B_1 - \sqrt{\frac{m\exp(\gamma t_1)}{\hbar}} y_{11}\right) d_\mu(\omega_1). \tag{30}$$

Taking the *T*-transform of equation (30) solves the quantum mechanical propagator of a particle in a damped harmonic oscillator potential. Following the calculation in (Bernido et.al, 2012; Baybayon et.al, 2019; Cubero et.al, 2010), the quantum mechanical propagator of a damped harmonic oscillator is given by

$$= \sqrt{\frac{m\, \Omega_{1\gamma} \exp\left(\frac{\gamma t_1}{2}\right)}{2\pi i\hbar \sin \Omega_{1\gamma t_1}}} \exp\left[\frac{im\Omega_{1\gamma}}{2\hbar \tan \Omega_{1\gamma t_1}} y_{11}^2 \exp(\gamma t_1)\right] \exp\left[-\frac{im\gamma}{4\hbar} y_{11}^2 \exp(\gamma t \tag{31}$$

Equation (31) agrees with the result of (Cubero et.al, 2010; Pepore, et.al, 2006; Grosche et.al, 1998).

The propagator of a damped coupled harmonic oscillator

In this section, the full form of our system's propagator which is the damped coupled harmonic oscillator will be presented. Since the Lagrangian of the damped coupled harmonic oscillator is separable into equation (9) and equation (10), the full propagator of the system can be solved by taking the product of the individual propagator of the decoupled oscillator (Pabalay et.al, 2007; Butanas et.al, 2016). Hence,

$$K = K_1 K_2 \tag{32}$$

where K_2 is of the same form as in equation (31) but with subscript 2 instead of 1. Thus, the propagator is equal to

$$K = \frac{m}{2\pi i\hbar}\sqrt{\frac{\Omega_{1\gamma}\Omega_{2\gamma}\exp\left(\frac{\gamma t_1}{2}\right)}{\sin\Omega_{1\gamma t_1}\sin\Omega_{2\gamma t_2}}}\exp\left[\left(\frac{im\Omega_{1\gamma}}{2\hbar\tan\Omega_{1\gamma t_1}}-\frac{im\gamma}{4\hbar}\right)y_1^2\exp(\gamma t_1)\right] \\ \times\exp\left[\left(\frac{im\Omega_{2\gamma}}{2\hbar\tan\Omega_{2\gamma t_1}}-\frac{im\gamma}{4\hbar}\right)y_2^2\exp(\gamma t_1)\right]. \tag{33}$$

Reversing the decoupling process from the variable $y_1 \to x_1$ in equation (2), the full propagator of the damped coupled harmonic oscillator is given by

$$\sqrt{\frac{\Omega_{1\gamma}\Omega_{2\gamma}\exp\left(\frac{\gamma t_1}{2}\right)}{\sin\Omega_{1\gamma t_1}\sin\Omega_{2\gamma t_2}}}\exp\left[\left(\frac{im\Omega_{1\gamma}}{4\hbar\tan\Omega_{1\gamma t_1}}-\frac{im\gamma}{8\hbar}\right)(x_1-x_2)^2 \\ \times\exp\left[\left(\frac{im\Omega_{2\gamma}}{4\hbar\tan\Omega_{2\gamma t_1}}-\frac{im\gamma}{8\hbar}\right)(x_1+x_2)^2\exp(\gamma \tag{34}$$

Now, turning off the damping factor γ, equation (34) reduces to the propagator of a coupled harmonic oscillator given by (Pabalay et.al, 2007)

$$_{HO} = \frac{m}{2\pi i\hbar}\sqrt{\frac{\Omega_1\Omega_2}{\sin\Omega_{1t_1}\sin\Omega_{2t_2}}}\exp\left[\left(\frac{im\Omega_1}{4\hbar\tan\Omega_{1\gamma t_1}}\right)(x_1-x_2)^2 \\ \times\exp\left[\left(\frac{im\Omega_2}{4\hbar\tan\Omega_{2t_1}}\right)(x_1+x_2)^2\right]. \tag{35}$$

THE TIME-DEPENDENT WAVEFUNCTION

In this section, the method in calculating the quantum wave function of a damped coupled harmonic oscillator will be presented. The propagator can be represented in terms of the time-dependent wavefunction, given by

$$K(0,x;t_1) = \sum_{n_1,n_2\in N}\Psi^*_{n_1n_2}(0,0)\,\Psi_{n_1n_2}(x_1,x_2;t_1). \tag{36}$$

Now, defining the following quantities

$$\sin\Omega_{l\gamma}t_1 = \frac{1}{2i}\left(\frac{1-z_l^2}{z}\right) \tag{37}$$

$$\cos\Omega_{l\gamma}t_1 = \frac{1}{2}\left(\frac{1+z_l^2}{z}\right) \tag{38}$$

$$z_l = \exp(-i\Omega_{l\gamma}t_1) \tag{39}$$

where l=1 and 2. Using these definitions, one can rewrite equation (34) as

$$K_{DCHO} = \frac{m}{\pi\hbar}\exp\left(\frac{\gamma t_1}{4}\right)(\Omega_{1\gamma}\Omega_{2\gamma}z_1z_2)^{\frac{1}{2}}(1-z_1^2)^{-\frac{1}{2}}(1-z_2^2)^{-\frac{1}{2}} \\ \times\exp\left[-\frac{m\Omega_{1\gamma}}{4\hbar}(x_1-x_2)^2\exp(\gamma t_1)\left(\frac{1+z_1^2}{1-z_1^2}\right)\right] \\ \times\exp\left[-\frac{m\Omega_{2\gamma}}{4\hbar}(x_1+x_2)^2\exp(\gamma t_1)\left(\frac{1+z_2^2}{1-z_2^2}\right)\right] \\ \times\exp\left[-\frac{im\gamma}{8\hbar}(x_1-x_2)^2\exp(\gamma t_1)\right] \\ \times\exp\left[-\frac{im\gamma}{8\hbar}(x_1+x_2)^2\exp(\gamma t_1)\right]. \tag{40}$$

Using Mehler's formula given by

$$(1-z^2)^{-\frac{1}{2}}\exp\left[\frac{4xyz-(x^2+y^2)(1+z^2)}{2(1-z^2)}\right] = \exp\left(-\frac{x^2+y^2}{2}\right)\sum_{n=0}^{\infty}\frac{\left(\frac{z}{2}\right)^n}{n!}H_n(x)H_n(y), \tag{41}$$

and with

$$x = 0 \tag{42}$$

$$y_l = \left(\frac{m\Omega_{l\gamma}}{2\hbar}\right)^{\frac{1}{2}}(x_1-x_2)\exp\left(\frac{\gamma t_1}{2}\right), \tag{43}$$

one can see that

$$(1-z_l^2)^{-\frac{1}{2}}\exp\left[-\frac{y^2}{2}\left(\frac{1+z_l^2}{1-z_l^2}\right)\right] \\ = (1-z_l^2)^{-\frac{1}{2}}\exp\left[-\frac{m\Omega_{l\gamma}}{4\hbar}(x_1-x_2)^2\exp(\gamma t_1)\left(\frac{1+z_l^2}{1-z_l^2}\right)\right] \\ = \sum_{n_l=0}^{\infty}\frac{\left(\frac{z_l}{2}\right)^{n_l}}{n_l!}H_{n_l}\left(\left(\frac{m\Omega_{l\gamma}}{2\hbar}\right)^{\frac{1}{2}}(x_1-x_2)\exp\left(\frac{\gamma t_1}{2}\right)\right) \\ \times\exp\left[-\frac{m\Omega_{l\gamma}}{4\hbar}(x_1-x_2)^2\exp(\gamma t_1)\right]. \tag{44}$$

for l=1 and 2. Incorporating equations (37),(38),(39), and (44) to equation (40), yields the following equation given by

$$K_{DCHO} = \exp\left(\frac{\gamma t_1}{4}\right)\exp\left[-\frac{im\gamma\exp(\gamma t_1)}{8\hbar}((x_1-x_2)^2+(x_1+x_2)^2)\right] \\ \times\exp\left[-\frac{m\exp(\gamma t_1)}{4\hbar}(\Omega_{1\gamma}(x_1-x_2)^2+\Omega_{2\gamma}(x_1+x_2)^2)\right] \\ \sum_{n_1n_2\in N}\frac{m(\Omega_{1\gamma}\Omega_{2\gamma})^{\frac{1}{2}}}{2^{n_1+n_2}\pi\hbar n_1!\,n_2!}\exp\left[-it_1\left(\frac{\Omega_{1\gamma}+\Omega_{2\gamma}}{2}+\Omega_{1\gamma}n_1+\Omega_{2\gamma}n_2\right)\right] \\ \times H_{n_1}\left(\sqrt{\frac{m\Omega_{1\gamma}}{2\hbar}}(x_1-x_2)\exp\left(\frac{\gamma t_1}{2}\right)\right) \\ \times H_{n_2}\left(\sqrt{\frac{m\Omega_{2\gamma}}{2\hbar}}(x_1-x_2)\exp\left(\frac{\gamma t_1}{2}\right)\right), \tag{45}$$

where H_n is the Hermite polynomial given by

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}. \tag{46}$$

Some of the first few Hermite polynomials are

$$H_0(x) = 1 \tag{47}$$

$$H_1(x) = 2x \tag{48}$$

$$H_2(x) = 4x^2 - 2 \quad (49)$$

$$H_3(x) = 8x^3 - 12x \quad (50)$$

Finally, comparing equation (45) to equation (36), one can extract the time-dependent wave function of the damped coupled harmonic oscillator given by

$$\Psi_{n_1,n_2} = \left(\frac{m(\Omega_{1\gamma}\Omega_{2\gamma})^{\frac{1}{2}}}{2^{n_1+n_2}\pi\hbar n_1!\, n_2!}\right)^{\frac{1}{2}} \exp\left(\frac{\gamma t_1}{4}\right) \exp\left[-\frac{im\gamma\exp(\gamma t_1)}{8\hbar}((x_1-x_2)^2+(x_1+x_2)^2)\right]$$
$$\times \exp\left[-\frac{m\exp(\gamma t_1)}{4\hbar}(\Omega_{1\gamma}(x_1-x_2)^2+\Omega_{2\gamma}(x_1+x_2)^2)\right]$$
$$\times \exp\left[-it_1\left(\frac{\Omega_{1\gamma}+\Omega_{2\gamma}}{2}+\Omega_{1\gamma}n_1+\Omega_{2\gamma}n_2\right)\right]$$
$$\times H_{n_1}\left(\sqrt{\frac{m\Omega_{1\gamma}}{2\hbar}}(x_1-x_2)\exp\left(\frac{\gamma t_1}{2}\right)\right) H_{n_2}\left(\sqrt{\frac{m\Omega_{2\gamma}}{2\hbar}}(x_1-x_2)\exp\left(\frac{\gamma t_1}{2}\right)\right). \quad (51)$$

The ground state wave function

In this section, the ground state wave function of a damped coupled harmonic oscillator is presented. This ground state wave function corresponds to the lowest energy state of the system. We set $n_1=n_2=0$, then equation (51) becomes

$$\Psi_{0,0} = \left(\frac{m(\Omega_{1\gamma}\Omega_{2\gamma})^{\frac{1}{2}}}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(\frac{\gamma t_1}{4}\right) \exp\left[-\frac{m\exp(\gamma t_1)}{4\hbar}(\Omega_{1\gamma}(x_1-x_2)^2+\Omega_{2\gamma}(x_1+x_2)^2)\right]$$
$$\times \exp\left[-\frac{im\gamma\exp(\gamma t_1)}{8\hbar}((x_1-x_2)^2+(x_1+x_2)^2)\right] \exp\left[-it_1\left(\frac{\Omega_{1\gamma}+\Omega_{2\gamma}}{2}\right)\right]. \quad (52)$$

One can write equation (52) in terms of its real and imaginary parts using the identity

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Then, it can be written as

$$\Psi_{0,0} = Re[\Psi_{0,0}] + i\, Im[\Psi_{0,0}]$$

where

$$Re[\Psi_{0,0}] = \left(\frac{m(\Omega_{1\gamma}\Omega_{2\gamma})^{\frac{1}{2}}}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(\frac{\gamma t_1}{4}\right) \exp\left[-\frac{m\exp(\gamma t_1)}{4\hbar}(\Omega_{1\gamma}(x_1-x_2)^2+\Omega_{2\gamma}(x_1 \right. \quad (53)$$
$$\times \cos\left[\frac{\Omega_{1\gamma}+\Omega_{2\gamma}}{2}t_1+\frac{m\gamma\exp(\gamma t_1)}{8\hbar}((x_1-x_2)^2+(x_1+x_2)^2)\right]$$

and

$$Im[\Psi_{0,0}] = \left(\frac{m(\Omega_{1\gamma}\Omega_{2\gamma})^{\frac{1}{2}}}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(\frac{\gamma t_1}{4}\right) \exp\left[-\frac{m\exp(\gamma t_1)}{4\hbar}(\Omega_{1\gamma}(x_1-x_2)^2+\Omega_{2\gamma}(x_1+x_2)^2)\right] \quad (54)$$
$$\times -\sin\left[\frac{\Omega_{1\gamma}+\Omega_{2\gamma}}{2}t_1+\frac{m\gamma\exp(\gamma t_1)}{8\hbar}((x_1-x_2)^2+(x_1+x_2)^2)\right],$$

respectively. Figure 1 shows the graph of the value of the ground state wavefunction of a damped coupled harmonic oscillator at a point where $x_1=0$ and $x_2=1$. The authors vary the damping coefficient, γ, from 0 to 0.9 and set the values of frequencies, $\Omega_1=0.75$ and $\Omega_2=0.50$. It shows in the figure that for a system with damping, the amplitude of the wavefunction decreases as you increase the time. Damped coupled harmonic oscillator is a dissipative system that loses energy from time to time. The decrease of the amplitude is due to the energy lost by the system. Also as you increase the value of the damping coefficient γ, the rate of of the decrease of the amplitude is increasing. The larger the value of the damping, the shorter the time needed to stop the motion of the object.

CONCLUSION

To conclude, the authors have shown the white-noise analysis as an effective tool in evaluating the propagators of quantum systems. Here, the authors studied the quantum dynamics of a damped coupled harmonic oscillator wherein friction is added to a coupled harmonic oscillator system. The authors also derived the time-independent wave function directly from the obtained propagator. The time –dependence of the wavefunction arises from the fact that our system is dissipative and it implies that the mechanical energy is not conserved. Lastly, this study shows the flexibility of the Hida-Streit approach in the field of complex systems.

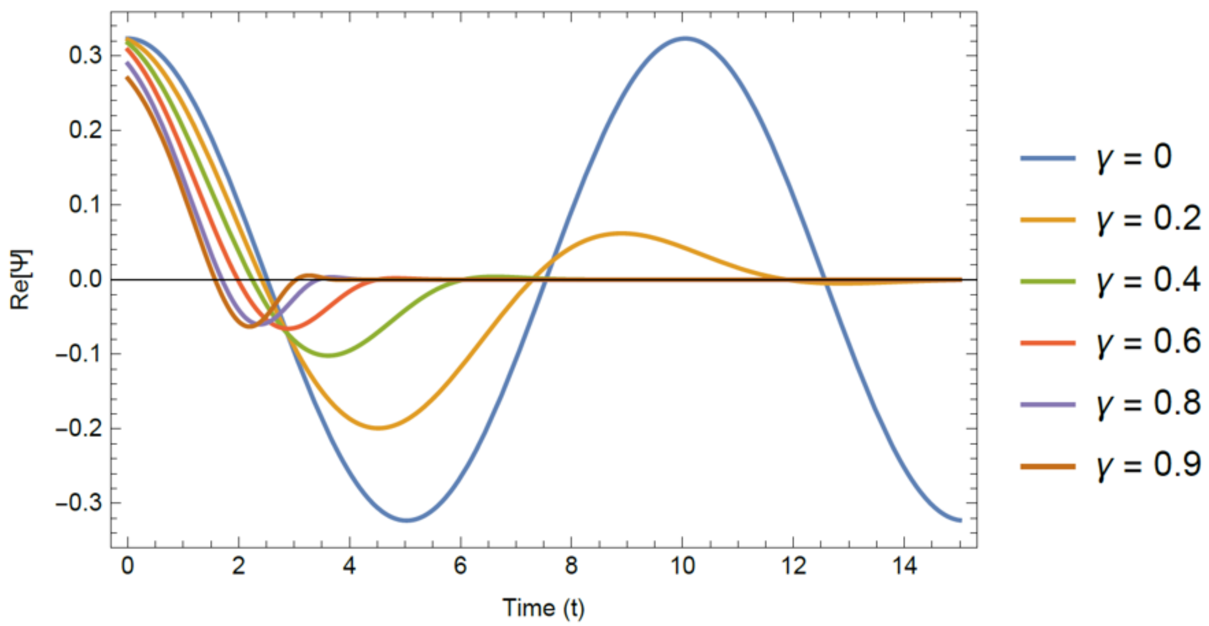

Figure 1. The 2D plot of the ground state wavefunction

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